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Quark Description of Hadronic Phases

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Abstract

We extend our proposal that major universality classes of hadronic matter can be understood, and in favorable cases calculated, directly in the microscopic quark variables, to allow for splitting between strange and light quark masses. A surprisingly simple but apparently viable picture emerges, featuring essentially three phases, distinguished by whether strangeness is conserved (standard nuclear matter), conserved modulo two (hypernuclear matter), or locked to color (color flavor locking). These are separated by sharp phase transitions. There is also, potentially, a quark phase matching hadronic K-condensation. The smallness of the secondary gap in two-flavor color superconductivity corresponds to the disparity between the primary dynamical energy scales of QCD and the much smaller energy scales of nuclear physics.

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In principle QCD ought to be an adequate basis for the description of hadronic matter in extreme conditions, including extremely high densities such as might be achieved deep in neutron star interiors, or transiently during collapse of massive stars, collisions between neutron stars, and laboratory heavy ion collisions. In practice however our ability to solve the equations is quite limited, and it has been challenging to provide a description firmly rooted in the microscopic theory. Specifically, the relationship between widely used phenomenological models, based on hadron degrees of freedom, and models based on quark degrees of freedom, themselves typically employing highly idealized abstractions of real QCD dynamics, has been quite unclear. It has generally been supposed that at relatively low density the hadronic description is appropriate, while at higher density some sort of quark description is appropriate.

Recently we argued for the radical proposal that in a slightly idealized version of QCD, with three degenerate flavors (and ignoring electromagnetism) there need be no sharp distinction between these pictures [1]. More precisely, we argued that the color-flavor locked state [2] (which can now, thanks to [3], be pretty rigorously identified as the correct ground state at asymptotically large densities), though firmly based on quark and gluon degrees of freedom, exhibits all the main features one might expect from naive extrapolation of phenomenologically-based conventional wisdom for the low-density phase. Specifically, the ground state exhibits confinement and chiral symmetry breaking, while the elementary excitations carry quantum numbers of pseudoscalar mesons, vector mesons, and baryons matching "phenomenological" expectations, notably including integral electric charges. The simplest hypothesis, then, is that there is only one phase. This implies, in particular, that color-flavor locking provides a rigorously defined, asymptotically controlled framework wherein the classic qualitative barriers to relating the microscopic (quark-gluon) to the macroscopic (hadron) degrees of freedom in QCD have been overcome.

It is obviously of great interest to extend this picture to real QCD, wherein the disparity between strange and non-strange masses is far from negligible. That is what we will sketch out here.

Our basic technique to understand QCD at high density is to assume weak coupling tentatively, and to work out the consequences of this assumption. Indeed, high density implies large Fermi surfaces, and therefore large momenta for an important class of low-energy degrees of freedom, namely the quasiparticle excitations near the Fermi surface. Their generic interactions will be characterized by large momentum transfer, and one might anticipate that asymptotic freedom can be invoked to analyze these interactions perturbatively. In the normal groundstate, however, there are two kinds of charged low-energy degrees of freedom – particle-hole excitations near the Fermi surface, and elementary gluons. These are dangerous, because their exchanges generate infrared divergences, which invalidate a straightforward perturbative approach.

Fortunately, we can do better. Indeed, as we know from the theory of superconductivity [4] and of He^3 superfluidity [5], arbitrarily weak attractive interactions near the Fermi surface drive a pairing instability. Early applications of these ideas to QCD are summarized in [6]. In favorable cases, including QCD with three or more degenerate flavors [1], interaction with the resulting condensate opens up gaps in all charged channels. Then there are no infrared problems, and the weak-coupling analysis is internally consistent. Of course this successful weak-coupling approach to high-density QCD is not the straightforward perturbative one: it is heavily rooted in asymptotic freedom and BCS theory, and includes both local and global spontaneous symmetry breaking (confinement and chiral symmetry breaking), derived from first principles.

I. ONSETS AND MISMATCHES

Until further notice we shall consider the case where the chemical potentials for all three flavors are set equal, and we shall pretend that electromagnetism has been turned off.

The preferred color-flavor locking ground state is particularly symmetric and energetically favorable for 3 degenerate flavors. What happens as we turn up the strange quark mass?

There are two main qualitative effects associated with the non-zero strange quark mass. Both can be identified either in a hadron picture or in a quark picture.

The first effect is that there are now two onset transitions. From the point of view of hadrons, one occurs when the chemical potential for baryon number exceeds the minimal energy/baryon in ordinary (nonstrange) nuclear matter, and the other when the chemical potential is large enough that strange baryons are also produced.

On the quark side, for free quarks, one would likewise expect two onsets, occurring when the chemical potential exceeds first the light and then the strange quark masses. Taking into account interactions, the situation is more complicated, though the qualitative outcome is very likely the same. The complication is that the onset transition is expected to (and, from a phenomenological point of view, had better) occur at a significantly larger value of the chemical potential, characteristic of nucleon rather than light quark masses. In particular, it should occur at a finite value of the chemical potential even for massless quarks. The possibility, for interacting massless quarks, of a first-order phase transition from the "void" state with chiral symmetry broken to a color superconducting state at finite density was raised within 2-flavor QCD in [7], [8]. It is implicit in the construction of the MIT bag model.

The second effect is that the light and strange Fermi surfaces will no longer be of equal size. Let us discuss this in more detail on the quark side. When the mismatch is much smaller than the gap one calculates assuming degenerate quarks, we might expect that it has very little consequence, since at this level the original particle and hole states near the Fermi surface are mixed up anyway. On the other hand, when the mismatch is much larger than the nominal gap, we might expect that the ordering one would obtain for degenerate quarks is disrupted, and that to a first approximation one can treat the light and heavy quark dynamics separately.

In particular, for weak coupling and Fermi surface mismatch, one should not expect mixed nonstrange-strange condensation. Indeed, unlike when one had matched Fermi surfaces, in this case one cannot form pairs of equal and opposite momenta, that the elastic scatterings

allowed in Fermi liquid theory connect. So there is not a large space of degenerate states connected by the relevant parts of the Hamiltonian, a weak perturbation has small effects. Thus, in the mixed channel one does not find Cooper instabilities at weak coupling.

One way to see this in a more quantitative fashion is to study a a schematic gap equation that describes the spin singlet pairing of two fermions with different masses. In a basis of particles of the first kind and holes of the second the quadratic part of the action is

$$\begin{pmatrix} \psi^\dagger_{(1)} & \psi_{(2)} \end{pmatrix} \begin{pmatrix} p_0 - \epsilon^1_p & \Delta \\ \Delta^* & p_0 + \epsilon^2_p \end{pmatrix} \begin{pmatrix} \psi_{(1)} \\ \psi^\dagger_{(2)} \end{pmatrix} . \tag{1}$$

Here, $\epsilon^{1,2}_p = E^{1,2}_p - \mu$ and $E^{1,2}_p = \sqrt{p^2 + m^2_{1,2}}$ where $m_{1,2}$ are the masses of particle one and two. The particle and hole propagators are determined by the inverse of the matrix (1). The off-diagonal (anomalous) propagator is

$$\frac{\Delta}{(p_0 - \epsilon^1_p)(p_0 + \epsilon^2_p) - \Delta^2}. \tag{2}$$

We study the effect of a zero range interaction $G(\psi_1\sigma_2\psi_2)(\psi^\dagger_1\sigma_2\psi^\dagger_2)$. The pairing is described by the gap equation

$$\Delta = G \int \frac{d^4p}{(2\pi)^4} \frac{\Delta}{(p_0 + R + i\delta \mathrm{sgn}(p_0))^2 - \bar{\epsilon}^2_p - \Delta^2}. \tag{3}$$

Here, we have introduced $\bar{\epsilon}_p = \bar{E}_p - \mu = (\epsilon^1_p + \epsilon^2_p)/2$ and $R = (\epsilon^1_p - \epsilon^2_p)/2$. In practice, we are interested in pairing between almost massless up or down quarks and massive strange quarks. In that case, $R \simeq m^2_s/(4p_F) \simeq m^2_s/(4\mu)$. The poles of the anomalous propagator are located at $p_0 = -R \pm \sqrt{\bar{\epsilon}^2_p + \Delta^2} - i\mathrm{sgn}(p_0)$. As usual, we close the integration contour in the lower half plane. Let us denote the solution of the gap equation in the case $R = 0$ by Δ_0. Then, if $R < \Delta_0$, the pole with the positive sign of the square root is always included in the integration contour and we have

$$\Delta = \frac{G\mu^2}{4\pi^2} \int d\bar{\epsilon}_p \frac{\Delta}{\sqrt{\bar{\epsilon}^2_p + \Delta^2}}. \tag{4}$$

This result is, up to a small correction in the density of states that we have neglected here, identical to the gap equation for degenerate fermions, so $\Delta \approx \Delta_0$. If, on the other hand,

$R > \Delta_0$ there only is a pole in the lower half plane if $\bar{\epsilon}_p > \sqrt{R^2 - \Delta^2}$. Carrying out the p_0 integration again leads to the gap equation (4), but with the $\bar{\epsilon}_p$ integration restricted by the condition just mentioned. This cuts out the infrared singularity at $\bar{\epsilon}_p = 0$ and one can easily verify that the gap equation does not have a non-trivial solution for weak coupling. We thus conclude that a necessary condition for pairing is that

$$m_s^2 < 4p_F\Delta(\mu). \tag{5}$$

So far, we have only dealt with a simple pair condensate involving strange and non-strange quarks. In practice, we are interested in a somewhat more complicated situation. In particular, we want to consider the transition between the color-flavor locked phase for small m_s and the two-flavor color superconductor in the limit of large m_s. This analysis can be carried out along the same lines as the toy model discussed above. We now consider the following free action

$$\begin{pmatrix} \psi^\dagger & \psi \end{pmatrix} \begin{pmatrix} (p_0 - \epsilon_p)\mathbb{1} - 2RM_s & \Delta_{ud}M_{ud} + \Delta_{us}M_{us} \\ \Delta_{ud}M_{ud} + \Delta_{us}M_{us} & (p_0 + \epsilon_p)\mathbb{1} + 2RM_s \end{pmatrix} \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix}, \tag{6}$$

where ψ is now a 9 component color-flavor spinor. M_s, M_{ud} and M_{us} are color-flavor matrices

$$M_s = \delta^{\alpha\beta}\delta_{a3}\delta_{b3} \tag{7}$$

$$M_{ud} = \epsilon^{3\alpha\beta}\epsilon_{3ab}, \tag{8}$$

$$M_{us} = \epsilon^{2\alpha\beta}\epsilon_{2ab} + \epsilon^{1\alpha\beta}\epsilon_{1ab}, \tag{9}$$

where α, β are color, and a, b flavor indices. Δ_{ud} is the gap for $\langle ud\rangle$ condensation, and Δ_{us} is the gap for $\langle us\rangle = \langle ds\rangle$ condensation. Color-flavor locking corresponds to the case $\Delta_{ud} = \Delta_{us}$, and the two flavor superconductor corresponds to $\Delta_{us} = 0$, $\Delta_{ud} \neq 0$. Flavor symmetry breaking is again caused by $R \simeq m_s^2/(4p_F)$. The 18×18 matrix (6) can be diagonalized exactly. The eigenvalues and their degeneracies are

$$
\begin{aligned}
&p_0 \pm \left(\epsilon_p^2 + \Delta_{ud}^2\right)^{1/2}, && d=3 \\
&p_0 - R \pm \left(\bar\epsilon_p^2 + \Delta_{us}^2\right)^{1/2}, && d=2 \\
&p_0 + R \pm \left(\bar\epsilon_p^2 + \Delta_{us}^2\right)^{1/2}, && d=2 \\
&p_0 \pm \left(\epsilon_p^2 + 2R\epsilon_p + 2R^2 + 2\Delta_{us}^2 + \tfrac{1}{2}\Delta_{ud}^2 - \tfrac{1}{2}S\right)^{1/2}, && d=1 \\
&p_0 \pm \left(\epsilon_p^2 + 2R\epsilon_p + 2R^2 + 2\Delta_{us}^2 + \tfrac{1}{2}\Delta_{ud}^2 + \tfrac{1}{2}S\right)^{1/2}, && d=1
\end{aligned}
\tag{10}
$$

where

$$
S = \left(8\Delta_{us}^2\left(\Delta_{ud}^2 + 4R^2\right) + \left(\Delta_{ud}^2 - 4R(\epsilon_p + R)\right)^2\right)^{1/2}. \tag{11}
$$

The result becomes easier to understand if we consider some simple limits. If we ignore flavor symmetry breaking, $R = 0$, and set $\Delta_{ud} = \Delta_{us}$ we find 8 eigenvalues $p_0 \pm (\epsilon_p^2 + \Delta^2)^{1/2}$ and one eigenvalue with the gap 2Δ. This is indeed the expected spectrum in the color-flavor locked phase. If, on the other hand, we set $\Delta_{us} = 0$ we find 4 eigenvalues $p_0 \pm (\epsilon_p^2 + \Delta^2)^{1/2}$ while the other 5 eigenvalues have vanishing gaps. Again, this is as expected.

We note that in the presence of flavor symmetry breaking the first three eigenvalues, which depend on Δ_{ud} only, are completely unaffected. For the next 4 eigenvalues, which only depend on Δ_{us}, the energy p_0 is effectively shifted by R. This is exactly as in the simple toy model discussed above. It implies that for $R > \Delta_{us}$, when we close the integration contour in the complex p_0 plane, we do not pick up this pole. The last two eigenvalues are more complicated. They depend on both Δ_{ud} and Δ_{us}, and they explicitly contain the flavor symmetry breaking parameter R. Nevertheless, the structure of the eigenvalues is certainly suggestive of the idea that for $R < \Delta_{us}^0$ we have $\Delta_{us} \simeq \Delta_{ud}$, and the gaps are almost independent of R, while at $R \simeq \Delta_{us}^0$ there is a discontinuity and Δ_{us} goes to zero.

This is borne out by a more detailed calculation. For this purpose, we add a flavor and color anti-symmetric short range interaction

$$
\mathcal{L} = \frac{K}{4}(\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma})(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \left(\psi_a^\alpha \sigma_2 \psi_b^\beta\right)\left(\psi_c^{\gamma\,\dagger}\sigma_2\psi_d^{\delta\,\dagger}\right). \tag{12}
$$

The free energy of the system is the sum of the quasi-particle contribution from (10) and the mean field potential $V = 1/K \cdot (\Delta_{ud}^2 + 2\Delta_{us}^2)$. There are two coupled gap equations,

which can be derived by varying the free energy with respect to the two parameters Δ_{ud} and Δ_{us}. Numerically, however, it is much simpler to minimize the free energy directly. As an example we show results for $\mu = 0.5$ GeV and $\Delta^0_{us} = 25$ MeV in Figure 1. We observe that the flavor symmetry breaking difference $\Delta_{ud} - \Delta_{us}$ is very small all the way up to the critical strange quark mass. At the critical mass, there is a discontinuous transition to a phase where Δ_{us} vanishes exactly. The value of the critical mass is very close to the estimate $m_s = 2\sqrt{\mu\Delta^0_{us}}$. We should note that in the case of color-flavor locking there are two physical gap parameters, the octet and the singlet gap. The scale for the critical strange quark mass is set by the octet gap.

II. TWO FLAVORS

When m_s is effectively large according to the preceding criterion, it becomes legitimate, when analyzing the light two flavors, to neglect the influence of the strange quark. This implies that the two-flavor analysis carried out in [7,8] is appropriate. The major result of that analysis is the existence of a condensate of the form

$$\langle q^{\alpha}_{L\,a} C q^{\beta}_{L\,b}\rangle = -\langle q^{\alpha}_{R\,a} C q^{\beta}_{R\,b}\rangle = \Delta\epsilon^{\alpha\beta 3}\epsilon_{ab}. \tag{13}$$

This condensate imparts a very substantial gap, plausibly of order 100 MeV at several times nuclear density, to two of the quark colors. It leaves an $SU(2)$ subgroup of color $SU(3)$, acting among these two colors of quarks, unbroken. The third color of quark does not acquire a gap from the primary condensate. As we will discuss below, these quarks can acquire a small gap from secondary modes of condensation. This gap, however, will not break the residual gauge symmetry, since it only involves quarks of the third color. The residual gauge symmetry can be broken in the case of three flavors, even if the strange quark is heavy, as long as the chemical potential is above the onset for strangeness. We will discuss possible modes of strangeness condensation below.

The expected low energy degrees of freedom are quasiparticle excitations around the Fermi surfaces of the third color quarks. It is important to note that in these channels,

there are no massless gluons. These low energy degrees of freedom are truly weakly coupled. Gluons in the unbroken $SU(2)$ remain massless, but their interactions become strong. We expect this means that the $SU(2)$ is confined, just as it would be in a vacuum theory with massive quarks. Of course, here we cannot claim rigorous control.

The primary condensate (13) leaves a residual gauge and chiral symmetry, as well as an unbroken $U(1)$ baryon number symmetry. This can be seen as follows: The condensate breaks both color hypercharge $Q_8 = \mathrm{diag}(1/3, 1/3, -2/3)$ and baryon number $B = \mathrm{diag}(1/3, 1/3)$, but leaves the combination $B' = B - Q_8$ invariant. Similarly, the condensate violates electric charge $Q_{em} = \mathrm{diag}(2/3, -1/3)$, but there is a modified charge operator $Q'_{em} = Q_{em} - Q_8$ which leaves the condensate invariant. So despite initial appearances the primary condensate is not an electric superconductor. A modified photon remains massless.

Under the unbroken symmetries, the quarks of the third color carry baryon number one. In addition to that, they are singlets under the residual $SU(2)$ gauge symmetry. And they carry charges (1,0) (in electron charge units) with respect to the modified photon. Thus, at asymptotically large density, the low energy – quark – degrees of freedom have the quantum numbers of the proton and the neutron!

At the level of the primary condensate, there is no gap for these modes. It is interesting to note, however, that the possible modes of secondary condensation involving these modes correspond exactly to the known possibilities for pairing in nuclear matter. This is the case because the remaining quarks not only carry the same isospin and charge quantum numbers as the proton and neutron but also, as emphasized above, they are both singlets under the residual gauge group. The superfluid pairings most often considered in the context of nuclear matter have the quantum numbers ${}^{2s+1}L_J = {}^1S_0$, ${}^3S_1$ and ${}^3P_2$. While ${}^1S_0$ is expected to dominate at low density, ${}^3P_2$ pairing is likely to take over at high density. The simplest operators with these quantum numbers are

$$ {}^1S_0 \qquad \psi C\gamma_5\tau_2\vec{\tau}\psi \tag{14} $$

$$^3S_1 \qquad \psi C\gamma_5\vec{\gamma}\tau_2\vec{\tau}\psi, \quad \psi C\vec{\Sigma}\tau_2\psi \tag{15}$$

$$^3P_2 \qquad S\psi C\gamma_5\gamma_i\hat{k}_j\tau_2\psi, \ldots \tag{16}$$

Here, $\hat{k} = \vec{k}/|\vec{k}|$ is the unit momentum vector and S projects out the symmetric traceless part. Nothing essentially new is gained by considering operators that involve the coupling of spin and angular momentum, such as $\psi C\vec{\gamma}\hat{k}\psi$. These operators can be reduced using the equations of motion.

One gluon exchange is repulsive in all these channels, with the exception of the tensor operator $\psi C\vec{\Sigma}\tau_2\psi$. That is the channel that was considered in [7]. In this work, the gap in the tensor channel was estimated to be in the range of several up to 100 keV. Of course, this estimate is exponentially sensitive to poorly determined couplings.

For later purposes, let us note that each of these possible secondary condensates breaks either isospin or rotational symmetry.

It is remarkable that in a theory where the dominant scale, the gap of the first two color quarks, is on the order of 100 MeV, the mass gap of the light degrees of freedom comes out to be so small. This is clearly reminiscent of the situation in real-world QCD at small density. QCD dynamically generates masses on the order of several hundred MeV, but the binding energy of nuclear matter and the gap in superfluid nuclear matter are small, on the order of a few MeV. This disparity is among the most fundamental qualitative facts about nuclear physics that one would like to understand from a microscopic viewpoint. From all previous standpoints known to us, it results from a conspiracy. Since just such a disparity appears as a calculable feature of the high-density phase, we are motivated to consider whether it is legitimate to extrapolate from the high-density phase to nuclear density. If it is, then no conspiracy need be invoked.

To justify such extrapolation, however, we must assume that there is no significant phase transition separating the high-density color superconducting phase from nuclear matter. (Changes in the nature of the small secondary condensate, which are certainly expected in view of the discussion above, are not germane here.) This has the startling implication, that

chiral symmetry must be – up to the effects of non-zero light quark masses, electromagnetism, and small secondary condensations – restored in nuclear matter. Whether this is true in reality, is worthy of much further investigation. It is suggestive, in this regard, that several experimental determinations suggest that the axial vector coupling g_A, whose free-nucleon value 1.26 reflects chiral symmetry breaking [9], is renormalized to very nearly unity in nuclear matter [10] [11] [12].

III. ONE FLAVOR

Above the onset for net strangeness, there is a Fermi surface for strange quarks. For large m_s there will be a big Fermi surface mismatch, and the strange quark will not pair with light quarks, as we discussed earlier. Then we can analyze its condensation independently. We expect a purely strange diquark condensate to form. This condensate breaks strangeness, but conserves a Z_2 that flips the sign of the strange quark.

Possible modes of strange quark superfluidity were studied in detail by Bailin and Love [6]. If the condensate is to be color antisymmetric, overall symmetry requires that it cannot be a scalar. A particularly interesting possibility is to lock color and spin

$$\langle s^\alpha C \gamma_i s^\beta \rangle = \Delta \epsilon^{\alpha\beta\gamma} \delta_i^\gamma . \tag{17}$$

With this condensate, color symmetry is completely broken, as is naive rotational symmetry, but a modified rotation symmetry involving simultaneous color and naive spatial rotation remains valid.

By taking the non-relativistic limit, and ignoring the complications due to antiparticles, we can vastly simplify the analysis of this phase. We consider the action

$$\begin{pmatrix} \psi^\dagger & \psi \end{pmatrix} \begin{pmatrix} p_0 - \epsilon_p & \Delta_{csl} M_{csl} \\ \Delta^*_{csl} M^*_{csl} & p_0 + \epsilon_p \end{pmatrix} \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix} , \tag{18}$$

where ψ is now a six-dimensional spin-color spinor. The pairing matrix M_{csl} corresponding to color-spin locking is given by

$$M_{csl} = (\sigma_2\sigma_k)_{ij}\epsilon^{k\alpha\beta}. \tag{19}$$

Other possible forms of order are

$$\begin{aligned} M_{pol} &= (\sigma_2\sigma_3)_{ij}\epsilon^{3\alpha\beta} && \text{polar} \\ M_A &= (\sigma_2\sigma_+)_{ij}\epsilon^{3\alpha\beta} && \mathrm{A-phase} \\ M_{pla} &= (\sigma_2\sigma_1)_{ij}\epsilon^{1\alpha\beta} + (\sigma_2\sigma_2)_{ij}\epsilon^{2\alpha\beta} && \text{planar} \end{aligned} \tag{20}$$

Let us first look at the color-spin locked phase. The quadratic action (18) is easily diagonalized. We find four eigenvalues with gap $\sqrt{2}\Delta_{csl}$ and two eigenvalues with gap Δ_{csl}. The degeneracies reflect the unbroken rotational symmetry. The state with gap $\sqrt{2}\Delta_{csl}$ has spin 3/2 and degeneracy $2s+1=4$, while the state with gap Δ_{csl} has spin 1/2 and degeneracy $2s+1=2$. In the polar phase, there are four states with gap Δ_{pol} while two states remain gapless. In the A-phase, only two quarks acquire a gap Δ_A, while the remaining four are gapless. In the planar phase, finally, all the physical excitations are gapless.

Since the color-spin locked phase is the only phase where all quarks acquire a gap it seems reasonable to expect that this is the groundstate of one flavor QCD at zero temperature and large chemical potential. This is what Bailin and Love found by analysing the Landau-Ginzburg functional, which however is only justified in the vicinity of the critical temperature. For simple model interactions, we find that either color-spin locking or the polar phase might be favored at zero temperature. A rigorous treatment is a feasible and interesting project, but will not be attempted here.

It is amusing to note that the color-spin locked phase has excitations with spin 3/2. This, of course, is the spin of the baryon in QCD with one flavor. The ordering of the states, however, appears to be inverted as compared to the expectation from the hadronic phase: while the ground state baryon has spin 3/2 in the hadronic phase, and the spin 1/2 baryon is an excited state, a spin 1/2 excitation is lowest the two in the high density phase. This might indicate that the high-density and nuclear phases must be separated by a phase transition, though strictly speaking it need not, since the ordering of low-lying levels is a non-universal feature. In any case, the situation is certainly less straightforward than for

three degenerate flavors.

IV. TWO PLUS ONE FLAVORS

We have suggested the existence of three major types of ordered phases for high-density QCD with 2+1 flavors: two-flavor color superconductivity with or without an accompanying strange superfluid, and color-flavor locking. Our description of each differs substantially, and our modelling above (and immediately below) suggests they are separated by sharp phase transitions, but the question arises whether the need for such transitions is guaranteed by any mismatch of physical symmetry. (Differences in gauge symmetry, which are meaningful only in the context of weak coupling, cannot be invoked here [13] [14].)

Fortunately, it is. The two-flavor color superconducting state, given only the primary condensation, has the global symmetry $SU(2)_L \times SU(2)_R \times U_S(1) \times \tilde{U}_B(1)$ for massless light quarks, where the first two factors are chiral isospin, the third is ordinary strangeness, and the fourth is (modified) baryon number. With non-zero but degenerate light quark masses, the first two factors collapse into a diagonal $SU(2)$. The strange superconducting state differs from this by breaking $U_S(1) \to Z_2$. The color-flavor locked state has $\tilde{SU}(2) \times \tilde{U}(1)$. These clearly all differ. However in the two-flavor superconducting state, since the primary condensation leaves gapless modes, we must in addition consider the effect of possible secondary condensations. However, as we already mentioned in passing, these condensations break either isospin or rotational symmetry (but not strangeness), so they cannot reproduce the symmetry of either of the other two major phases.

Our discussion of the phase structure of dense matter in QCD with two light and one intermediate mass flavor can be illustrated concretely with the help of a simple schematic model. Let us consider a short range interaction with the quantum numbers of one gluon exchange

$$\mathcal{L} = K(\bar{\psi}\gamma_\mu\lambda^a\psi)(\bar{\psi}\gamma^\mu\lambda^a\psi), \tag{21}$$

characterized by a coupling constant K. We study the phase structure of the system in

the mean field approximation. Since we are interested in chiral symmetry breaking and dynamical mass generation as well as superfluidity we can no longer restrict ourselves to particle-hole pairs, but have to include anti-particle (hole) contributions also. The free energy is of the form

$$F = \sum_i \epsilon(\sigma_i, \delta_i) + V, \tag{22}$$

$$\epsilon(\sigma, \delta) = \int \frac{d^3p}{(2\pi)^3} \left(\sqrt{(E_p - \mu)^2 + \delta^2} + \sqrt{(E_p + \mu)^2 + \delta^2} \right), \tag{23}$$

where $E_p = \sqrt{p^2 + (\sigma + m)^2}$ is the single particle dispersion relation and V is the mean field potential. The integration over p is regularized by a sharp cutoff Λ.

We consider the following phases:

1) A phase with chiral symmetry breaking only. This phase is characterized by a chiral condensate

$$\langle \bar{q}_a^\alpha q_b^\beta \rangle = \delta^{\alpha\beta}(\delta_{ab}\Sigma_0 + \delta_{a3}\delta_{b3}\Sigma_s). \tag{24}$$

2) The color flavor locked phase, including the possibility of a dynamically generated contribution to the strange quark mass

$$\langle q_a^\alpha C\gamma_5 q_b^\beta \rangle = (\delta_a^\alpha \delta_b^\beta \Delta_1 + \delta_b^\alpha \delta_a^\beta \Delta_2), \tag{25}$$

$$\langle \bar{s}^\alpha s^\beta \rangle \ \ = \delta^{\alpha\beta}\Sigma_s \tag{26}$$

3) The superconducting phase, with or without strange quark condensation

$$\langle q_a^\alpha C\gamma_5 q_b^\beta \rangle = \epsilon^{\alpha\beta 3}\epsilon_{ab3}\Delta_{ud} \tag{27}$$

$$\langle \bar{s}^\alpha s^\beta \rangle \ \ = \delta^{\alpha\beta}\Sigma_s. \tag{28}$$

In practice we ignore the condensation energy of the light quarks of the third color and the strange quark superfluid, since both of them are expected to be quite small.

The effective potential and the mass gaps in the three different phases are easily calculated. We find

$$
\begin{aligned}
1) \qquad & \sigma_1 = \ldots = \sigma_6 = \frac{16}{3} K \Sigma_0 \\
& \sigma_7 = \ldots = \sigma_9 = \frac{16}{3} K (\Sigma_0 + \Sigma_s) \\
& V = K \left(48\Sigma_0^2 + 32\Sigma_0\Sigma_s + 16\Sigma_s^2 \right) \\
2) \qquad & \delta_1 = \ldots = \delta_8 = \frac{K}{3}\left(3\Delta_1 - \Delta_2\right) \qquad \delta_9 = \frac{K}{3} 8\Delta_2 \\
& \sigma_7 = \ldots = \sigma_9 = \frac{16}{3} K \Sigma_s \\
& V = K \left(-16\Delta_1\Delta_2 + 16\Sigma_s^2 \right) \\
3) \qquad & \delta_1 = \ldots = \delta_4 = \frac{4}{3} K \Delta_{ud} \\
& \sigma_7 = \ldots = \sigma_9 = \frac{16}{3} K \Sigma_s \\
& V = K \left(\frac{16}{3} \Delta_{ud}^2 + 16\Sigma_s^2 \right)
\end{aligned}
\tag{29}
$$

In the color-flavor locked phase we include the shift in the eigenvalues due to the strange quark mass, following our detailed discussion in the first section. In particular, the integral in the non-strange-strange particle-hole channel is restricted to $\bar{\epsilon}_p^2 > R^2 + \delta^2$. The only difference is that we allow for the possibility of a dynamically generated contribution to the strange quark mass. It is now straightforward to minimize the free energy in the three phases separately and to determine the global minimum by comparing the different solutions. The result is shown in Figure 2. We have chosen $\Lambda = 600$ MeV and fixed $K = 32\Lambda^{-2}$ in order to reproduce a $\mu = 0$ quark constituent mass of 400 MeV.

We note that each of the phases disussed above is indeed favorable for some values of the chemical potential and strange quark mass. The transition from the color-flavor locked phase to the nuclear or hypernuclear phase is shifted to smaller strange quark masses as compared to the calculation in section II. This effect is due to the large dynamically generated strange quark mass. Also, as a function of the chemical potential, there is an intermediate nuclear phase between the vacuum and color-flavor locked phases even for very small strange quark mass. Again, this is due to the large dynamically generated strange quark mass. We also note that even for small strange quark masses $m_s \sim 50$ MeV, the onset for strangeness is significantly above the onset for light quarks. This, of course, is the most direct manifestation

of a dynamically generated strange quark mass.

We should stress that the model is clearly very crude, and in particular that the locations of the phase boundaries should not be regarded as quantitative predictions. Nevertheless, we believe that the existence of the different phases, and the topology of the phase diagram are robust features consequences of the physical mechanisms we have discussed.

There is an interesting consequence, in the strange superconductor phase, of the interplay between color-spin locking for the strange quarks and color symmetry breaking induced by the primary light quark condensation. The preferred color axis induces a preferred spin axis, so that the material becomes a strange ferromagnet, in particular violating even the modified rotation symmetry.

Finally, we would like briefly to discuss two interesting possibilities for high density that have been proposed and much debated in the literature, from the point of view (*i.e.*, working down from the asymptotics) used here. One is the hypothesis of stable strange quark matter [15]. In our language, it is the hypothesis that in the interacting theory there is no separate onset transition for strangeness, but strange particles appear immediately and abundantly at the "void to matter" transition. In our Figure 2, it would mean that the lower portion of the sc region would be squeezed out. Although in the nature of things we cannot rule this out, since it is proposed as a strong-coupling phenomenon and our methods are intrinsically weak-coupling, nothing we have encountered suggests it. The second is K-condensation [16]. It is of course perfectly trivial to write down bilinear quark-antiquark condensates that match the hadronic K-condensation quantum numbers (namely, $\langle \bar{s}\gamma_5 u\rangle$ or $\langle \bar{s}\gamma_0\gamma_5 u\rangle$). Only slightly less trivial is that one can do so with the residual ungapped degrees of freedom, after the primary condensation in two-flavor superconductivity. Again, however, nothing we have encountered suggests this is a very favorable possibility.

Acknowledgement

Related issues are addressed in the paper "Unlocking Color and Flavor in Superconducting Strange Quark Matter", by M. Alford, J. Berges, and K. Rajagopal, MIT preprint MIT-CTP-2844, appearing simultaneously. Where they overlap, our conclusions agree. We

thank these authors for showing us their work prior to publication, and for informative discussions.

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FIGURES

FIG. 1. Light quark gap Δ_{ud} and strange-non-strange gap Δ_{us} as a function of the strange quark mass.

FIG. 2. Phase diagram calculated in the schematic model described in Section IV. We show the phase boundaries between the vacuum phase (void), the color-flavor locked phase (cfl), the nuclear phase (sc), and the hypernuclear phase (scs). Note that the transition line from the nuclear phase to the color-flavor locked phase turns sharply towards the void-cfl transition at very small m_s. For $m_s = 0$ there is a direct transition from the vacuum phase to the color-flavor locked phase. Also note that the phase diagram was calculated with a $\Lambda = 600$ MeV form factor, so the turnover of the cfl-scs transition near $\mu = 600$ MeV is an artefact of this formfactor.

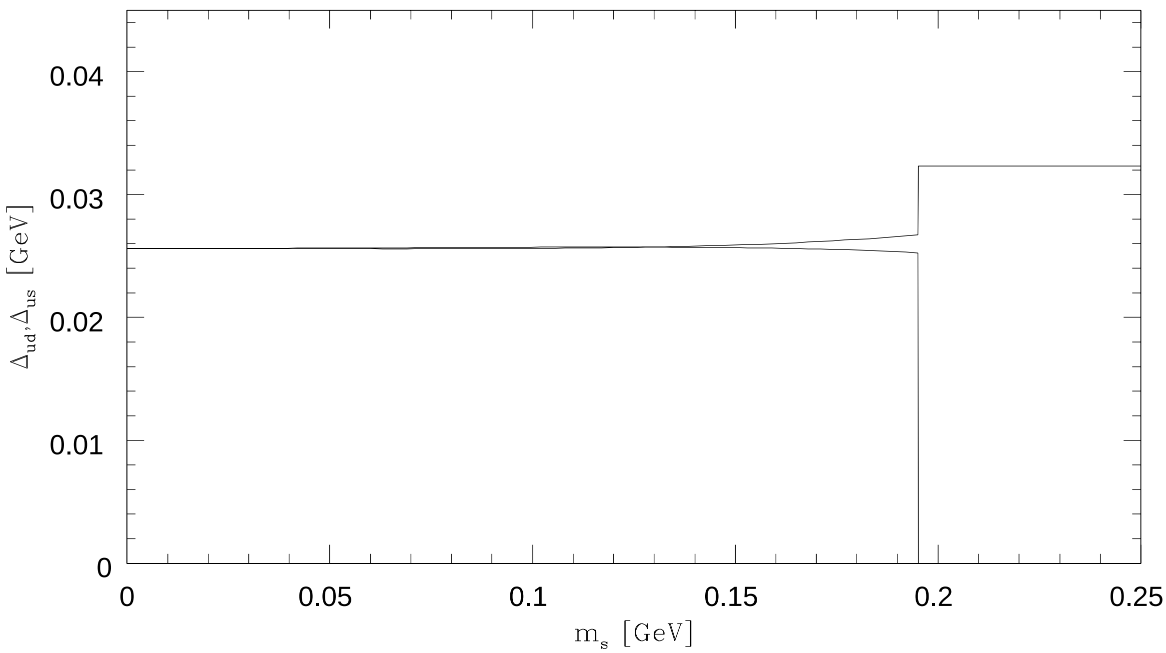

FIG. 1.

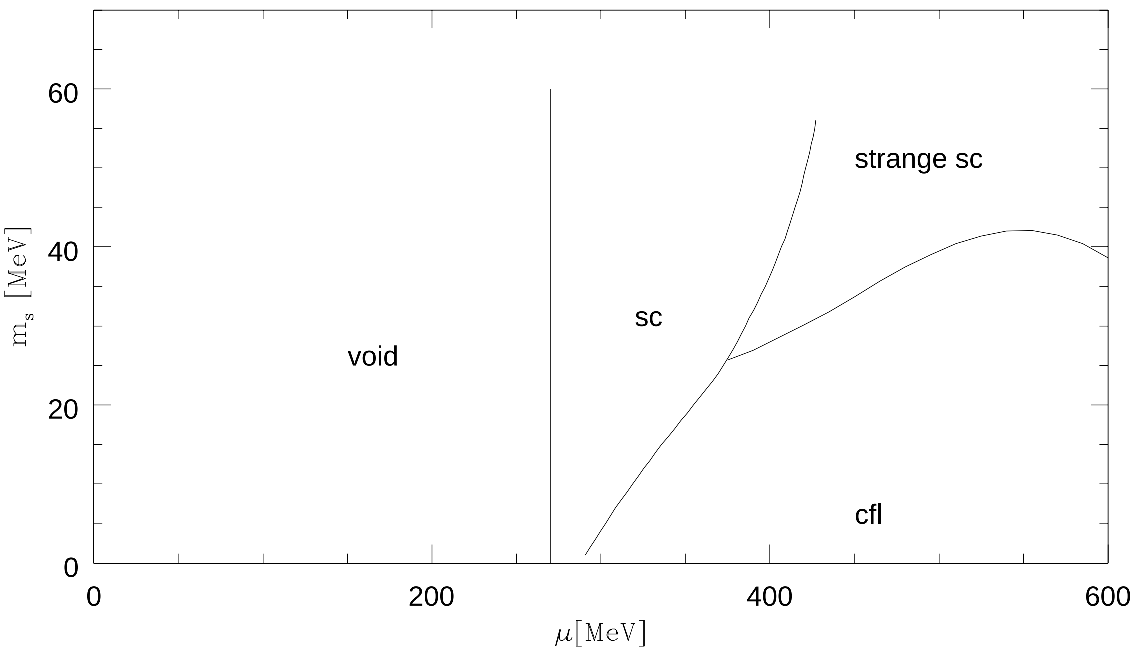

FIG. 2.